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Spherically Symmetric Perfect Fluid Solutions of Einstein's Equations in Noncomoving Coordinates

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Almost all known spherically symmetric perfect fluid solutions of Einstein's equations have been obtained in comoving coordinates and nearly all are shear-free. In this paper we study two solutions in noncomoving coordinates and show that they contain shear.

1. INTRODUCTION

The most common method of solving Einstein's equations for perfect fluids is to use comoving coordinates. In the case of spherical symmetry, with which we are concerned in this paper, the method is very successful for pressure-free matter (dust) for which the general solution was found by Tolman (1934). For matter with pressure, however, the solutions obtained by the comoving method are almost exclusively free of shear (Kramer *et al.*, 1980). It would be very interesting to know more about the effect of shear on relativistic fluid flow.

A pioneering paper on perfect fluid metrics in noncomoving coordinates was that of McVittie and Wiltshire (1977). They found several classes of metrics in these coordinates, but did not investigate their physical behavior, and in particular did not ascertain whether they have shear. Another work in which noncomoving coordinates were used was that of Biech and Das (1990), who found two shearing solutions, one contained among McVittie and Wiltshire's, and one not. Biech and Das also give a good summary of the few spherically symmetric flows with pressure which are shearing. Other recent work on shearing flows is that of Collins (1991), Herrera *et al.* (1991), and Kitamura (1989).

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In this paper we take two of McVittie and Wiltshire's solutions and study them in detail, proving that they have shear. We are able to show that the space-times contain regions in which the fluid satisfies physically reasonable energy conditions.

The plan of the paper is as follows. In Section 2 we give the field equations in a form suitable for noncomoving coordinates, and define the kinematical fluid parameters we shall use. The two exact solutions are studied in Sections 3 and 4, and there is a Conclusion.

2. FIELD EQUATIONS

In this section we follow closely McVittie and Wiltshire. We start with a general spherically symmetric time-dependent metric in the form

$$ds^2 = -e^{2\mu}\,d\xi^2 - r^2\,d\Omega^2 + e^{2\lambda}\,d\eta^2$$
$$d\Omega^2 := d\theta^2 + \sin^2\theta\,d\phi^2 \tag{2.1}$$

where μ, λ, and r are functions of ξ and η only, and we number the coordinates as follows:

$$x^1 = \xi, \qquad x^2 = \theta, \qquad x^3 = \phi, \qquad x^4 = \eta$$
$$-\infty < \xi < \infty, \qquad 0 \le \theta \le \pi, \qquad 0 \le \phi \le 2\pi, \qquad -\infty < \eta < \infty \tag{2.2}$$

The field equations are

$$G^i_k := R^i_k - \tfrac{1}{2}\delta^i_k R = -8\pi T^i_k \tag{2.3}$$

The energy tensor for a perfect fluid is

$$T^i_k = (p+\rho)u^i u_k - \delta^i_k p \tag{2.4}$$

ρ and p are the proper density and pressure, respectively, and u^i is the unit four-velocity satisfying

$$g_{ik}u^i u^k = 1 \tag{2.5}$$

Substituting the components g_{ik} of the metric (2.1) into the left-hand side of (2.3), we find

$$G^1_2 = G^1_3 = G^2_3 = G^4_2 = 0$$

so from (2.4), $u_2 = u_3 = 0$ and the nonvanishing components of T^i_k are

$$T^1_1 = (\rho+p)u^1 u_1 - p$$
$$T^2_2 = T^3_3 = -p \tag{2.6}$$
$$T^4_4 = (\rho+p)u^4 u_4 - p$$

$$T^1_4 = -e^{2(\lambda-\mu)}T^4_1 = (\rho+p)u^1 u_4 \tag{2.7}$$

By means of (2.5) we can eliminate ρ, p, and the velocity components from (2.6) and (2.7), obtaining

$$e^{2\lambda}(T_1^4)^2+e^{2\mu}(T_2^2-T_1^1)(T_2^2-T_4^4)=0 \tag{2.8}$$

and so, using (2.3), we get

$$e^{2\lambda}(G_1^4)^2+e^{2\mu}(G_2^2-G_1^1)(G_2^2-G_4^4)=0 \tag{2.9}$$

This is the only field equation if we allow noncomoving coordinates. The comoving solutions for the metric (2.1) are those solving (2.9) by

$$G_4^1=0, \qquad G_1^1=G_2^2 \tag{2.10}$$

which lead to $u^1=0$. In this paper we shall suppose

$$u^1\neq 0 \tag{2.11}$$

We can obtain ρ, p, and u^i in terms of T_k^i from (2.6):

$$\rho=T_1^1+T_4^4-T_2^2, \qquad p=-T_2^2 \tag{2.12}$$

$$e^{2\mu}(u^1)^2=\frac{T_2^2-T_1^1}{(T_1^1-T_2^2)+(T_4^4-T_2^2)} \tag{2.13}$$

$$e^{2\lambda}(u^4)^2=\frac{T_4^4-T_2^2}{(T_1^1-T_2^2)+(T_4^4-T_2^2)} \tag{2.14}$$

We shall take the sign of u^4 to be positive, and the sign of u^1 is then obtained from (2.7).

Of course, every flow of a single-component perfect fluid can be expressed in comoving coordinates (though there will be singularities where streamlines intersect), so it is possible, at least in principle, to write our solutions in comoving coordinates. Our motive here is to consider flows which are easier to obtain and study in noncomoving coordinates. However, we shall have occasion to consider the transformations to comoving coordinates $x^{i'}$. In space-times with metric (2.1) this is achieved by writing $x^{2'}=x^2$, $x^{3'}=x^3$, and finding $x^{1'}$, $x^{4'}$ such that

$$u^{1'}=\frac{\partial x^{1'}}{\partial x^1}u^1+\frac{\partial x^{4'}}{\partial x^4}u^4=0 \tag{2.15}$$

$$g^{1'4'}=-\frac{\partial x^{1'}}{\partial x^1}\frac{\partial x^{4'}}{\partial x^1}e^{-2\mu}+\frac{\partial x^{1'}}{\partial x^4}\frac{\partial x^{4'}}{\partial x^4}e^{-2\lambda}=0 \tag{2.16}$$

We are especially interested in flows with shear. The shear tensor σ_{ik} is given by

$$\sigma_{ik}=u_{(i;k)}-\dot{u}_{(i}u_{k)}-\tfrac{1}{3}h_{ik}\theta \tag{2.17}$$

where the acceleration $\dot{u}_i$ is

$$\dot{u}_i = u_{i;j}u^j \tag{2.18}$$

and

$$h_{ik} = g_{ik} - u_i u_k$$

Parentheses denote symmetrization, e.g.,

$$u_{(i;k)} = \tfrac{1}{2}(u_{i;k} + u_{k;i})$$

We shall also need the expansion scalar

$$\theta = u^i_{;i} \tag{2.19}$$

In the next two sections we study two metrics representing the shearing flow of a perfect fluid in noncomoving coordinates.

3. METRIC A

We present this in the form given by Ray (1978)

$$ds^2 = -e^{2(\alpha+\eta)}(d\xi^2 + d\Omega^2 - d\eta^2) \tag{3.1}$$

where $\alpha = \alpha(\xi)$ and to satisfy (2.9) we require

$$6\alpha'' - 2(\alpha')^2 + 3 = 0 \tag{3.2}$$

where a prime means $d/d\xi$. This metric is a special case of (9.15) in McVittie and Wiltshire (1977).

The physical quantities for (3.1) are found by calculating T^i_k from (2.3) and then using (2.12)–(2.14), taking into account (2.7):

$$8\pi p = \tfrac{1}{3}e^{-2(\alpha+\eta)}(5\omega - 6) \tag{3.3}$$

$$8\pi\rho = e^{-2(\alpha+\eta)}(5 - 3\omega) \tag{3.4}$$

$$u^1 = -2e^{-(\alpha+\eta)}\alpha' X^{-1/2}, \qquad u^4 = 3e^{-(\alpha+\eta)}X^{-1/2} \tag{3.5}$$

where $X := (9 - 4\omega)$, and $\omega := (\alpha')^2$; here and throughout the paper the positive square root is to be taken. For the solution to represent a fluid and to have physical significance we require

$$X > 0, \qquad \rho > 0, \qquad p \geq 0, \qquad \rho - p \geq 0 \tag{3.6}$$

These conditions are satisfied if

$$6/5 \leq \omega \leq 3/2 \tag{3.7}$$

so there exists a region of space-time in which the solution is physical provided ω lies in this range.

The general solution of (3.2) is

$$e^{-\alpha/3} = A\cosh\chi + B\sinh\chi, \qquad \chi = \xi/\sqrt{6} \tag{3.8}$$

where A, B are arbitrary constants. In the special case $A = \pm B$ this reduces to

$$\alpha = \varepsilon(3/2)^{1/2}\xi + k, \qquad \varepsilon = \pm 1 \tag{3.9}$$

where k is a constant. This gives

$$\omega = 3/2$$

corresponding to a stiff fluid with an equation of state $p = \rho$.

If $B^2 > A^2$, we find from (3.8) that $\omega > 3/2$, so (3.7) cannot be satisfied. If $A^2 > B^2$, we can write (3.8) as

$$e^{-\alpha/3} = C\cosh(\chi + \beta) \tag{3.10}$$

where C, β are new constants replacing A, B. From this we obtain

$$\omega = \tfrac{3}{2}\tanh^2(\chi + \beta)$$

so (3.7) requires

$$\tfrac{4}{5} \leq \tanh^2(\chi + \beta) \leq 1$$

which is satisfied for all η in two disjoint ranges of χ such that

$$\chi \geq q_1 \qquad \text{and} \qquad \chi \leq q_2 \qquad (q_1 > q_2)$$

where q_1, q_2 depend on β. Thus the condition (3.7) is fulfilled over part of the range $-\infty < \xi < \infty$, but not the whole range. It is further clear from (3.3) and (3.4) that p and ρ are infinite when $\eta = -\infty$ and also when $e^{-2\alpha} = +\infty$. We find from (3.10)

$$e^{-2\alpha} = C^6\cosh^6(\chi + \beta)$$

Hence a singularity in p and ρ occurs at $\xi = \pm\infty$. These values of ξ also correspond to the centers of spherical symmetry because, as we see from (3.1), the surface area of a 2-sphere with coordinate radius $\xi = \pm\infty$ is permanently zero. The model has a finite, two-centered spherical geometry, but ρ and p are infinite at the centers. There is no equation of state of the form $p = f(\rho)$.

The kinematical quantities for the perfect fluid flow represented by metric A, obtained by evaluating (2.17)-(2.19), are

$$\dot{u}_i = (-9\alpha' X^{-2}, 0, 0, -6\omega X^{-2})$$
$$\theta = 6e^{-(\alpha+\eta)}(2\omega-3)(2\omega-5)X^{-3/2}$$
$$\sigma_{11} = 18e^{(\alpha+\eta)}(2\omega-3)X^{-5/2}$$
$$\sigma_{14} = 12e^{(\alpha+\eta)}\alpha'(2\omega-3)X^{-5/2}$$
$$\sigma_{44} = 8e^{(\alpha+\eta)}\omega(2\omega-3)X^{-5/2}$$
$$\sigma_{22} = \sigma_{33}\operatorname{cosec}^2\theta = -e^{(\alpha+\eta)}(2\omega-3)X^{-3/2}$$

The shear invariant is given by

$$\sigma_{ab}\sigma^{ab} = 6e^{-2(\alpha+\eta)}(2\omega-3)^2 X^{-3}$$

Regarding these kinematical quantities, we note the following:

(i) The acceleration $\dot{u}_i$ is nonzero, thus showing that metric A represents a solution different from those of Biech and Das (1990), which have $\dot{u}_i = 0$. Our $\dot{u}_i$ is independent of time.

(ii) Excluding the special case $\omega = \frac{3}{2}$, we see that the expansion is positive in the physical range (3.7).

(iii) The shear is nonzero if $\omega \neq \frac{3}{2}$.

(iv) The case $\omega = \frac{3}{2}$ has zero expansion and shear; in fact it is static, as we shall shortly show.

Let us consider the solution A in comoving coordinates. Taking first the special case (3.9), we find that the transformation

$$\log R = \eta + \tfrac{1}{2}\varepsilon\sqrt{6}\,\xi + k, \qquad t = \sqrt{3}\,\eta + \varepsilon\sqrt{2}\,\xi$$

satisfies (2.15) and (2.16) and takes (3.1) into

$$ds^2 = -2\,dR^2 - R^2\,d\Omega^2 + R^2\,dt^2$$

which is static. It represents a static sphere of stiff fluid, but the pressure and density are infinite at $R = 0$. This solution is in fact contained in a class of solutions given by Ibañez and Sanz (1982).

In the case of (3.8), taking $e^{-\alpha/3}$ in the form (3.10), we find that the transformation to comoving diagonal coordinates [i.e., coordinates satisfying (2.15) and (2.16)] is

$$r = \eta - 3\log\sinh(\chi+\beta)$$
$$t = \eta - 2\log\cosh(\chi+\beta)$$

However, attempts to invert these formulas and to write the metric (3.1) in terms of r and t lead to extremely cumbersome expressions and it becomes clear why this solution has not been discovered in comoving coordinates.

4. METRIC B

This is

$$ds^2 = z^{4/3}\, d\eta^2 - z^{2/3}(d\xi^2 + \xi^2\, d\Omega^2) \tag{4.1}$$

where $z = a\xi^2 + b\eta$ and a, b are constants. The numbering and ranges of coordinates are as in (2.2), except that here $0 \le \xi < \infty$. It was given by McVittie and Wiltshire (1977) in a slightly different notation. Calculating T^i_k from (2.3) and using (2.12)-(2.14) and (2.7), we find

$$8\pi p = \tfrac{1}{9} z^{-8/3}(7b^2 z^{-2/3} + 16az + 20ab\eta)$$
$$8\pi\rho = \tfrac{1}{9} z^{-8/3}(3b^2 z^{-2/3} - 56az + 20ab\eta)$$
$$u^1 = 2\xi Y^{-1/2}, \qquad u^4 = -ba^{-1} z^{-2/3} Y^{-1/2}$$

where $Y := b^2 a^{-2} - 4\xi^2 z^{2/3}$ and, as usual, the positive square root is to be taken. For the metric (4.1) to represent a realistic perfect fluid, we require

$$Y > 0, \qquad \rho > 0, \qquad p \ge 0, \qquad \rho - p \ge 0 \tag{4.2}$$

One finds that these conditions can be satisfied in the neighborhood of $\xi = 0$ if $a < 0$, $b > 0$, and η is suitably chosen. Thus there exists a region of space-time in which the solution is physically reasonable.

The kinematical quantities of the fluid flow represented by metric B are as follows:

$$\dot{u}_i = (-4b^2 a^{-2} \xi z^{2/3} Y^{-2}, 0, 0, -8ba^{-1}\xi^2 z^{4/3} Y^{-2})$$
$$\theta = (a^{-1} z^{-4/3} W + 12) Y^{-1/2} - 4(3a)^{-1} \xi^2 z^{-2/3} W Y^{-3/2}$$
$$\sigma_{11} = 8b^2 (9a^3)^{-1} \xi^2 W Y^{-5/2}$$
$$\sigma_{14} = 16b(9a^2)^{-1} \xi^3 z^{2/3} W Y^{-5/2}$$
$$\sigma_{44} = 32(9a)^{-1} \xi^4 z^{4/3} W Y^{-5/2}$$
$$\sigma_{22} = \sigma_{33} \operatorname{cosec}^2\theta = -4(9a)^{-1} \xi^4 W Y^{-3/2}$$
$$\sigma_{ab}\sigma^{ab} = 32(27a^2)^{-1} \xi^4 z^{-4/3} W^2 Y^{-3}$$

where

$$W = -6az^{4/3} + 4a^2 \xi^2 z^{1/3} - b^2 z^{-1/3}$$

We notice that the pressure and density are infinite only on the hypersurface $z = 0$, which represents an expanding or contracting 2-sphere. However, there is another hypersurface, namely $Y = 0$, on which the kinematical quantities are singular. There is a zero-pressure surface $7b^2 z^{-2/3} + 16az + 20ab\eta = 0$, which suggests the possibility of using the solution as a finite spherical interior for a vacuum (Schwarzschild) exterior.

Comoving diagonalized coordinates (denoted here by r, t) require the solution of (2.15) and (2.16). A possible set is

$$t = z$$

with r satisfying the differential equation

$$2a\xi\frac{\partial r}{\partial \xi} - bz^{-2/3}\frac{\partial r}{\partial \eta} = 0$$

As in the case of metric A, we have not been able to write the solution explicitly in comoving coordinates.

5. CONCLUSION

We have examined in detail two solutions of Einstein's equations for a perfect fluid which were obtained by the use of noncomoving coordinates. Unlike the great majority of spherically symmetric perfect fluid solutions obtained in comoving coordinates, the ones considered here have shear. This suggests that noncomoving coordinates may be useful in the study of relativistic perfect fluid space-times.

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